

Thermodynamics of a black hole based on a generalized uncertainty principle

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ABSTRACT: We study thermodynamic quantities and the stability of a black hole in a cavity using the Euclidean action formalism by Gibbons and Hawking based on the generalized uncertainty relation which is extended in a symmetric way with respect to the space and momentum without loss of generality. Two parameters in the uncertainty relation affect the thermodynamical quantities such as energy, entropy, and the heat capacity. In particular, it can be shown that the small black hole is unstable and it may decay either into a minimal black hole or a large black hole. We discuss a constraint for a large black hole comparable to the size of the cavity in connection with the critical mass.

KEYWORDS: Black Holes, Models of Quantum Gravity.

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1. Introduction

Hawking’s quantum field theoretical analysis [1] has shown that a Schwarzschild black hole has a thermal radiation with a temperature $T_H = (8\pi M)^{-1}$, where M is the mass of the black hole. Subsequently, this issue has been investigated in the thermodynamical regime through the path-integral approach to the quantization of gravity [2, 3]. It has been also shown that the entropy of a black hole is always equal to one quarter of the area of the event horizon in fundamental units and a stationary system without event horizon has no entropy. Moreover, the thermodynamics in an asymptotically anti-de Sitter black hole has been studied in ref. [4]. Using the Euclidean action approach by Gibbons and Hawking [2], the thermodynamic local quantities such as temperature, energy, entropy, and surface pressure, have been evaluated in a cavity with a finite size [5–8]. Unlike other quantities related to the size of cavity, the entropy does not have local property of gravity since it agrees with Bekenstein-Hawking entropy depending only on the event horizon. It means that the entropy is independent of the asymptotic behavior of fields.

Now, the conventional Heisenberg uncertainty principle(HUP) has been promoted to the generalized uncertainty principle(GUP) [9–13] based on some aspects of quantum gravity and the string theory, which is given by

$$\Delta x \Delta p \geq \hbar \left(1 + \ell^2 \frac{(\Delta p)^2}{\hbar^2} \right), \tag{1.1}$$

where it leads to the minimal length of $\Delta x_{\min} = 2\ell$. The cutoff ℓ may be chosen as a string scale in the context of the perturbative string theory or Plank scale based on the quantum gravity. In the brick-wall method [14], the GUP has been used to calculate the entropy of black holes without a cutoff parameter [15–18] where the minimal length plays the role of ultraviolet cutoff and it is regarded as a natural cutoff. Also, the corrections to entropy by the GUP has been studied in other methods [26]. Recently, the thermodynamics and its

stability for the Schwarzschild black hole have been studied by applying the GUP [19–21]. They obtained a remnant after evaporation of a black hole and it may be stable, however, the relevant thermodynamic quantities should be treated as local quantities because the GUP effects significantly appear near horizon.

On the other hand, one can generalize the GUP by considering $(\Delta x)^2$ along with $(\Delta p)^2$ in the uncertainty relation (1.1) for the same footing [22]. Then, the symmetric generalized uncertainty principle(SGUP) can be written by

$$\Delta x \Delta p \geq \hbar \left(1 + \frac{(\Delta x)^2}{L^2} + \ell^2 \frac{(\Delta p)^2}{\hbar^2} \right), \quad (1.2)$$

which leads to the minimal length of $\Delta x_{\min} = 2\ell/\sqrt{1 - 4\ell^2/L^2}$ and the minimal momentum of $\Delta p_{\min} = 2\hbar/(L\sqrt{1 - 4\ell^2/L^2})$, where L is another uncertainty constant.

In this work, based on the SGUP, we would like to study the thermodynamic behaviors of physical quantities of a black hole in the Euclidean action formalism which gives the local Tolman temperature naturally, and investigate the stability of the black hole in terms of the heat capacity. When the temperature is over the critical temperature [23–25], the small black hole created by the phase transition is unstable and decays to hot flat space or grows to the cavity size, which can be expanded to the infinity. However, in the GUP improved thermodynamics, the small black hole cannot decay to hot flat space by the thermal radiation since there is the minimal size of a black hole. The difference from the previous works [19–21] mainly comes from the local Tolman temperature whereas the global temperature has been used in the thermodynamic analysis so far. So, in section 2, we shall obtain the local Tolman temperature in this SGUP and then calculate the thermodynamic local quantities compatible with the local temperature related to the size of a cavity. The local entropy, which is consistent with the thermodynamic first law, will be also derived. In section 3, we will take the limit of $L \rightarrow \infty$ called the GUP case and investigate the thermodynamics and the stability of the black hole which have not been discussed in earlier works. In section 4, when L is finite, thermodynamic analysis will be done. Finally, we draw some discussions in section 5.

2. Thermodynamic quantities in SGUP

The thermodynamic quantities will be defined in a cavity, which means that we have to consider the local temperature based on the SGUP. It seems to be plausible to consider the local temperature rather than the Hawking temperature in the cavity. For this purpose, we solve eq. (1.2) for the momentum uncertainty in terms of the position uncertainty [19]

$$\frac{\Delta p}{\hbar} = \frac{\Delta x}{2\ell^2} \left(1 \pm \sqrt{1 - \frac{4\ell^2}{L^2} - \frac{4\ell^2}{(\Delta x)^2}} \right), \quad (2.1)$$

and putting $\Delta x = 2M$, we identify the emitted photon energy with the black hole temperature up to a calibration factor so that

$$T_{\text{SGUP}} = \frac{M}{4\pi\ell^2} \left(1 - \sqrt{1 - \frac{4\ell^2}{L^2} - \frac{\ell^2}{M^2}} \right), \quad (2.2)$$

where we set $\hbar = G = 1$ for simplicity. We assume on the basis of thermodynamic consistency that the emitted photons have a thermal black body spectrum. The remainder of our work depends upon this assumption. If we consider a large black hole ($\ell/M \ll 1$) with $M \ll L$, then the modified temperature, in the leading order, goes to the well-known Hawking temperature when we choose the negative sign. So, the limiting case of $\ell \rightarrow 0$ and $L \rightarrow \infty$ is well-defined. If we set $L \lesssim M$, then it gives a correction to the Hawking temperature, $T_{\text{SGUP}} \approx 1/(8\pi M) + M/(2\pi L^2)$.

Now, the partition function can be written as $Z = \exp(-I) = \exp(-\beta F)$, where I , β , and F are the first-order Euclidean Einstein action, the inverse of temperature T , and the free energy of system in the cavity of the finite radius. The Euclidean action with a subtraction term is defined by

$$I = I_1 - I_0, \tag{2.3}$$

where

$$I_1 = -\frac{1}{16\pi} \int d^4x \sqrt{g} \mathcal{R} + \frac{1}{8\pi} \oint d^3x \sqrt{\gamma} K. \tag{2.4}$$

Here, K is the trace of the extrinsic curvature tensor K_{ij} of the boundary $S^1 \times S^2$ of $r = R = \text{const.}$ and γ_{ij} is its induced three-metric. The line element of a Schwarzschild black hole is

$$ds_E^2 = f d\tau^2 + f^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{2.5}$$

where $f(r) = 1 - 2M/r$ and τ is the Euclidean time. The period of the Euclidean time is $\beta_{\text{SGUP}} = T_{\text{SGUP}}^{-1}$, and then the proper length of the S^1 of the boundary is

$$\beta = T^{-1} = \int_0^{\beta_{\text{SGUP}}} d\tau \sqrt{g_{\tau\tau}} = \beta_{\text{SGUP}} \sqrt{f(R)}. \tag{2.6}$$

Since $\sqrt{\gamma}$ and K are explicitly calculated as $\sqrt{\gamma} = R^2 \sin \theta \sqrt{f(R)}$, $K = -2\sqrt{f(R)}/R - M/R^2 \sqrt{f(R)}$ at the boundary, and I_1 becomes

$$I_1 = \beta_{\text{SGUP}} \left(\frac{3}{2} M - R \right). \tag{2.7}$$

On the other hand, I_0 is evaluated for a flat four-metric with boundary $S^1 \times S^2$. In this case, the period of the Euclidean time is β and we have $\sqrt{\gamma} = R^2 \sin \theta$ and $K = -2/R$, which yields $I_0 = -\beta R$. This subtraction term normalizes the thermal energy to zero for the Schwarzschild geometry with $M = 0$ and has no effect on the other physical quantities. Combining these two terms, the Euclidean action becomes

$$I = \beta R + \frac{\beta}{\sqrt{f(R)}} \left(\frac{3}{2} M - R \right). \tag{2.8}$$

From eqs. (2.2) and (2.6), the local temperature measured on the boundary in a thermal equilibrium is

$$T = \frac{M}{4\pi\ell^2} \left(1 - \sqrt{1 - \frac{4\ell^2}{L^2} - \frac{\ell^2}{M^2}} \right) \left(1 - \frac{2M}{R} \right)^{-\frac{1}{2}}, \tag{2.9}$$

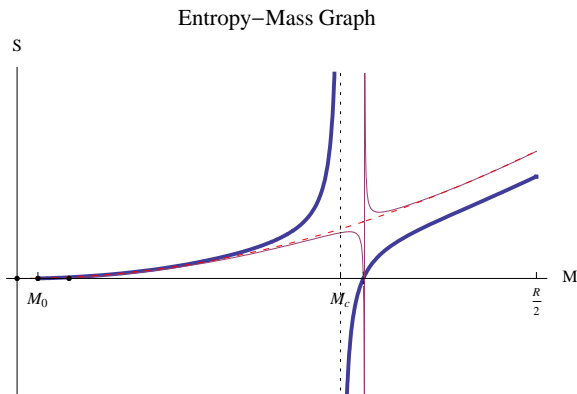


Figure 1: Three entropies are plotted for the case of $L = 20$ and $\ell = 1/5$ (SGUP; thick solid line), $L \rightarrow \infty$ and $\ell = 1/2$ (GUP; thin solid line), and $L \rightarrow \infty$ and $\ell = 0$ (HUP; dashed line). Each entropy becomes zero at the corresponding minimal masses $M_0 = \ell/\sqrt{1 - 4\ell^2/L^2}$, ℓ , and 0. The three points near the origin in the horizontal axis represent the minimal mass for HUP, SGUP, and GUP cases from the left, respectively.

which is nothing but the Tolman temperature with the redshift factor. The interesting point to distinguish from the conventional Tolman temperature is that M in eq. (2.9) is bounded and all its value lies between $\ell/\sqrt{1 - 4\ell^2/L^2} < M < R/2$. Even at the minimal black hole, the temperature is finite $T = 1/4\pi\ell\sqrt{1 - 4\ell^2/L^2}$ for $M = \ell/\sqrt{1 - 4\ell^2/L^2}$ while it is divergent for $M = 0$ in the conventional Tolman temperature.

Since the area of S^2 of the boundary is $A = 4\pi R^2$, the total thermodynamic internal energy within the boundary R becomes

$$E = \left(\frac{\partial I}{\partial \beta} \right)_A = R - R\sqrt{1 - \frac{2M}{R}} \left[\frac{1 - \varepsilon - r_0/R}{1 - \varepsilon - r_c/R} \right], \quad (2.10)$$

where $\varepsilon = 4(M/L)^2(1 - 4\ell^2/L^2)$, $r_0 = (3M/2)(1 - 4\ell^2/L^2 + \alpha)$, $r_c = M(2 - \varepsilon - 4\ell^2/L^2 + \alpha)$, and $\alpha = \sqrt{1 - 4\ell^2/L^2 - \ell^2/M^2}$. In the finite R , $E \approx R - R(1 - 3M/2R)/\sqrt{1 - 2M/R}$ for which M goes to $\ell/\sqrt{1 - 4\ell^2/L^2}$ and $E \approx 2M$ as M goes to $R/2$, respectively. What the nonvanishing black hole mass even in this minimal black hole means is that there is a positive definite smallest energy corresponding to a remnant. The thermodynamic energy is singular at $r_c = R(1 - \varepsilon)$ where this point corresponds to the critical mass $M_c \equiv M|_{\partial T/\partial M=0}$, on which the temperature goes to the critical temperature $T_c \equiv T|_{M=M_c}$. Note that below the critical temperature, no black hole exists.

From the free energy relation, $F = E - TS$, the black hole entropy is explicitly written as

$$S = \beta E - I = 2\pi M^2 \alpha \left(1 - \frac{4\ell^2}{L^2} + \alpha \right) \frac{1 - 3M/R}{1 - \varepsilon - r_c/R}, \quad (2.11)$$

where we used eqs. (2.8), (2.9), and (2.10). Note that it can be reduced to the well-known area law $S = 4\pi M^2$ for $\ell \rightarrow 0$ and $L \rightarrow \infty$ which is independent of the size of the cavity so that it suggests that the black hole entropy is independent of the asymptotic behavior

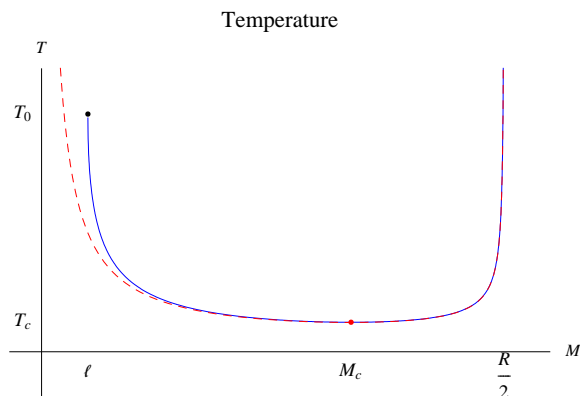


Figure 2: The solid line and the dotted line show the profiles of the temperature based on the GUP and the HUP, respectively. The crucial difference between them comes near the end state of the black hole, and the minimal length prevents the black hole from the total evaporation [$\ell = 1$, $R = 20$: $M_c \approx 6.69171$, $T_c \approx 0.010396$, $T_0 \approx 0.083882$].

of the gravitational field and matter fields. However, once the minimal length is assumed, then the entropy is related to not only the minimal length but also the boundary through the cavity size. Apparently, the area law is no longer hold, but it can be easily proved that the thermodynamic first law, $dE = TdS$, is automatically satisfied for fixed A . It is interesting to note that the entropy of the minimal black hole is zero which means that the minimal black hole state whose mass is $M = \ell/\sqrt{1 - 4\ell^2/L^2}$ can be a single state at the end of the black hole evaporation. We plotted the entropies which correspond to HUP, GUP, and SGUP in figure 1.

3. GUP case of $L \rightarrow \infty$

The thermodynamic stability of the black hole can be studied in terms of the heat capacity in the GUP limit. For this purpose, we consider a large L limit of $L \rightarrow \infty$, so that the temperature (2.9) is reduced to

$$T = \frac{M}{4\pi\ell^2} \left(1 - \sqrt{1 - \frac{\ell^2}{M^2}} \right) \left(1 - \frac{2M}{R} \right)^{-\frac{1}{2}}. \tag{3.1}$$

In the conventional analysis with a cavity, the behavior of temperature of a small black hole looks similar to that of the large black hole since the small black hole has the small horizon radius compared to the size of the cavity so that its temperature gets large as in the HUP case, while the large black hole which is comparable to the size of the cavity gives a high temperature due to the redshift factor since the local observer is almost near the horizon. However, in the present calculations based on the GUP, the black hole does not completely evaporate, in other words, the temperature of the small black hole is not so high because of the cutoff.

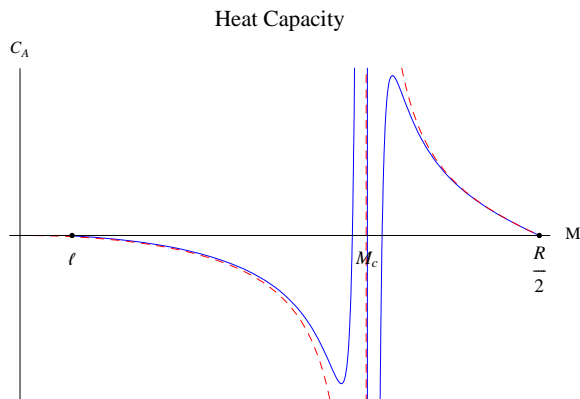


Figure 3: The dotted and the solid line show the behaviors of heat capacities based on the HUP and GUP, respectively. This figure is plotted for $\ell = 1$, $R = 20$: $M_c \approx 6.69171$, $C_A|_{M \rightarrow \ell} \approx -5.93412$.

As seen in figure 2, there are no black hole states for $T < T_c$, and both small and large black hole can exist within $T_c < T < T_0$ where $T_0 = T|_{M=\ell} = \left(4\pi\ell\sqrt{1-2\ell/R}\right)^{-1}$, while only the large black hole solution is possible for $T > T_0$. Note that there is a forbidden region between $0 < M < \ell$, which tells us that there is no small black hole whose mass is less than the minimal length dimension. On the other hand, it is possible to obtain the minimal black hole of $M = \ell$ in contrast to the conventional thermodynamical analysis.

To study the thermodynamic stability of a black hole, one can consider the heat capacity at a constant surface, which is defined by

$$\begin{aligned}
 C_A &\equiv \left(\frac{\partial E}{\partial T}\right)_A \\
 &= -2\pi M^2 \left(1 - \frac{r_c}{M}\right)^2 \left(1 - \frac{2M}{R}\right) \frac{(1 - r_-/R)(1 - r_+/R)}{(1 - r_c/R)^3}, \tag{3.2}
 \end{aligned}$$

where the constants are $r_{\pm} = a \pm \sqrt{a^2 - b}$, $a = (M^3/\ell^2)[1 + 3\ell^2/(2M^2) - (1 - \ell^2/M^2)^{3/2}]$, and $b = 9M^2[1 - \ell^2/(3M^2)]$, respectively. If we take the limit of $\ell \rightarrow 0$, it is naturally reduced to the result of the HUP [6],

$$C_A^{\text{HUP}} = -8\pi M^2 \left(1 - \frac{2M}{R}\right) \left(1 - \frac{3M}{R}\right)^{-1}, \tag{3.3}$$

which is positive for $R/3 < M < R/2$ while it is negative for $0 < M < R/3$. Moreover, it is singular at $M \rightarrow M_c = R/3$. In this case, the small black hole of $M \rightarrow 0$ is unstable to decay into either into pure thermal radiation or to a large black hole. In our generalized case as shown in figure 3, there are two different aspects from the HUP case. First, the critical behavior of the heat capacity near the critical mass is more complicated so that we can not obtain the definite criteria for the stability for $M < M_c$ or $M > M_c$. For $M < M_c (M > M_c)$, there may exist a stable(unstable) region near the critical mass while most part of the region is unstable (stable). Secondly, the minimal black hole exists in our analysis so that the black hole can not evaporate completely. For $M \rightarrow \ell$ in eq. (3.2), the

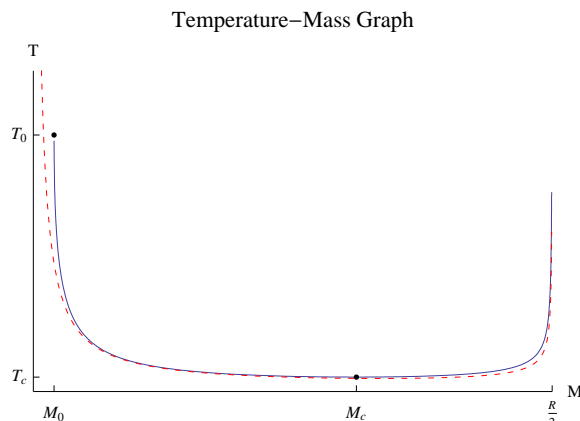


Figure 4: The solid line describes the behavior of the temperature for the finite L , which gives an additional correction mainly to the large black hole. Eventually, the temperature are shifted up due to the two uncertainty constants ℓ and L . [$L = 20$, $\ell = 1/5$, $R = 10$: $M_c \approx 3.11365$, $T_c \approx 0.0228477$, $M_0 \approx 0.20004$, $T_0 \approx 0.406175$].

heat capacity of the minimal black hole is $C_A \approx -2\pi\ell^2$, where the horizon of the black hole is the same with the minimal length, $r_H = 2M = 2\ell = \Delta x_{\min}$.

4. Another case of finite L

In this section, we are now in a position to study how the SGUP affects the thermodynamic quantities by assuming L to be finite, while, in the previous section, we have studied the GUP limit of $L \rightarrow \infty$. As seen in figure 4, the finite L gives a temperature correction mainly to the large mass black hole comparable to the cavity size, $M \sim R/2$. Note that the minimal black hole mass M_0 is larger than that of GUP so that it gives higher temperature.

We calculate the heat capacity at a constant surface to see whether a black hole in a cavity is stable or not, which is now given by

$$C_A = -2\pi M^2 \left(1 - \frac{4\ell^2}{L^2} + \alpha\right)^2 \left(1 - \frac{2M}{R}\right) \frac{(\mathcal{A} - 2\mathcal{B}/R + \mathcal{C}/R^2)}{(1 - \varepsilon - r_c/R)^3}, \quad (4.1)$$

where the constants are $\mathcal{A} = (1 - 4\ell^2/L^2)(1 + 4\alpha M^2/L^2)$, $\mathcal{B} = (M^3/\ell^2)[(1 - 4\ell^2/L^2)(1 - \varepsilon + 4(\ell/L)^2(1 + 5\alpha/2)) + (3/2 - 10\ell^2/L^2)\ell^2/M^2 - \alpha^3(1 - \varepsilon)]$, and $\mathcal{C} = 9M^2(1 - 4\ell^2/L^2 - \ell^2/3M^2)(1 + 4\alpha M^2/L^2)$, respectively. If we take the limit of $L \rightarrow \infty$, then it naturally recovers the heat capacity in eq. (3.2). On the other hand, as for the other extreme limit of $\ell \rightarrow 0$ keeping L to be finite, the heat capacity becomes

$$C_A = -8\pi M^2 \left(1 - \frac{2M}{R}\right) \left(\mathcal{A} - \frac{2\mathcal{B}}{R} + \frac{\mathcal{C}^2}{R^2}\right) \left[1 - \frac{4M^2}{L^2} - \frac{M}{R} \left(3 - \frac{4M^2}{L^2}\right)\right]^{-1}, \quad (4.2)$$

where $\mathcal{A} \rightarrow (1 + 4M^2/L^2)$, $\mathcal{B} \rightarrow M(3 + 10M^2/L^2 - 8M^4/L^4)$, and $\mathcal{C} \rightarrow 9M^2(1 + 4M^2/L^2)$. Of course, the heat capacity (3.3) of the HUP case can be derived for $L \rightarrow \infty$ and $\ell \rightarrow 0$.

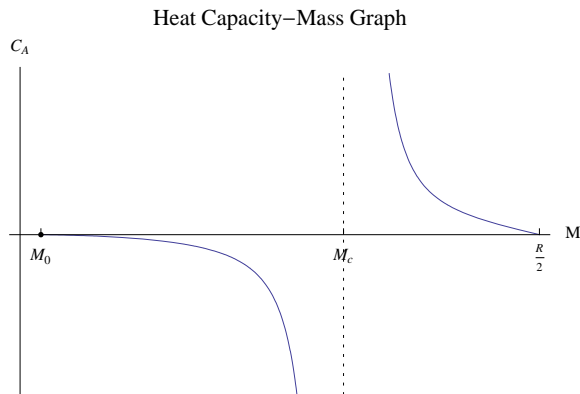


Figure 5: The heat capacity is singular at the critical mass and it is ill-defined when the mass is less than the minimal mass. It also shows that it is negative for $M < M_c$ while it is positive for $M > M_c$. This figure shows that it is critically different from that of the GUP case. The heat capacity goes to $C_A \approx -2\pi M_0^2(1 - 3M_0/R)/(1 - 2M_0/R)$ as $M \rightarrow M_0$ and zero as $M \rightarrow R/2$. This figure is plotted for $L = 20$, $\ell = 1/5$, $R = 10$: $M_c \approx 3.11365$, $M_0 \approx 0.20004$, $C_A|_{M \rightarrow M_0} \approx -0.174082$.

In figure 5, the complicated behavior of the heat capacity near the critical mass disappears and resembles the behavior of the heat capacity of HUP. Therefore, the whole profile of the heat capacity for the finite L case is very similar to that of the HUP case. In fact, the complicated behavior of the heat capacity near the critical mass appears again when both $L \gg 1/\ell$ and $L \gg R$ are satisfied. The exact critical condition exists but it is too lengthy to write down.

5. Discussion

We have studied thermodynamic quantities and the stability of the black hole in a cavity based on the extended GUP called symmetric GUP whose limits are well defined such as $L \rightarrow \infty$ and $\ell \rightarrow 0$. In particular, following the Euclidean action formalism in a cavity, the Tolman temperature has been used in deriving the heat capacity, and it gives the consistent thermodynamic first law along with the appropriate energy and the entropy. Moreover, this entropy can not be written in the form of the area, whereas it recovers the area law when we take the HUP limit of $L \rightarrow \infty$ and $\ell \rightarrow 0$. The entropy is usually proportional to the area since it is independent of the asymptotic behavior of the gravitational field and matter fields; however, it depends on two uncertainty parameters, L and ℓ in our case.

In the HUP case, a small black hole in unstable equilibrium over the critical temperature may decay either into pure thermal radiation or to a large black hole. In the GUP case, the off-shell free energy [23–25] can be defined by $F_{\text{offshell}} = E - TS$ where eqs. (2.10) and (2.11) are used with the arbitrary temperature. It shows that the small black hole in the unstable equilibrium may decay into the minimal black hole or a larger stable black hole, however, whose final mass should be smaller than the critical mass in figure 2. From the extrema condition of the free energy, $(dF_{\text{offshell}}/dM)_R = 0$, it can be shown that the

mass of the small unstable black hole can not exceed the critical mass, because the off-shell free energy (potential) is singular at the critical mass due to the definition of the GUP temperature of eq. (2.9). If the initial mass of the black hole is larger than the critical mass, then it may decay into a really large black hole whose mass is nearly $R/2$, which is also a big difference from the HUP case.

In fact, there remains further issues in connection with GUP(SGUP) calculations. First, the Hawking temperature modified by the GUP in ref. [19] should be confirmed by other independent methods. One of them may be a scattering method in connection with the grey-body factor. Second, the resulting modified temperature (2.2) subject to the GUP(SGUP) should be consistent with the metric giving the well-known Hawking temperature through the periodicity of the Euclideanized metric. It means that we have to consider the back reaction of the geometry properly. We hope these issues will be addressed elsewhere.

Acknowledgments

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